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VIABLE AND MONOTONE TRAJECTORIES OF DIFFERENTIAL INCLUSIONS.(U)
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VIABLE AND MONOTONE TRAJECTORIES OF DIFFERENTIAL INCLUSIONS

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ABSTRACT

This is an exposition of Haddad's theorems on the existence of solutions to differential inclusions $x' \in F(t,x)$, $x(t_0) = x_0$ that satisfy the "viability condition"

$$\forall\, t \in [0,T]\ ,\quad x(t) \in K(t)$$

where the subsets $K(t)$ are given. The existence theorem generalizes the Nagumo theorem when F is single-valued and when both F and K do not depend on t. Haddad's theorems provide necessary and sufficient conditions for a differential inclusion to have viable trajectories. They involve the concepts of Bouligand contingent cone to K when $K(t) = K$ does not depend on K and of contingent derivatives of the set-valued map $t \to K(t)$. We adapt these results to characterize the existence of monotone trajectories with respect to a preorder $\geq$:

$$\forall\, s > t\ ,\quad x(s) \geq x(t).$$

We conclude by presenting Saint-Pierre's theorem on the convergence of a generalized Newton method converging to a solution $u_\star \in K$ of the inclusion $0 \in F(x_\star)$ in a nondeterministic way. This is of upmost importance in economics where uncertainty plays a crucial role. It is also important in control theory (since the differential inclusion (5) can be regarded as involving implicitly controls) as well as for solving implicit differential equations $f(x,x') = 0$.

AMS(MOS) Subject Classification 47H10, 49B30

Key Words: Differential inclusions, invariant subsets, contingent cones, contingent derivatives, implicit differential equation.

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SIGNIFICANCE AND EXPLANATION

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This paper begins with an exposition of results of Haddad on the solvability of a differential inclusion (in which the rate of change $\dot{x}$ of the state variable x is restricted to lie in a specified set $F(x)$ and is not given exactly) for a solution x with $x(t) \in K$ for $t \geq 0$ and K a specified set of "viable states". Problems of this sort arise naturally in control theory and economics. A main point is the generality of the assumptions under which the theory is successful. Results on monotone trajectories and the convergence of a generalized Newton method to a rest state are also presented.

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VIABLE AND MONOTONE TRAJECTORIES OF DIFFERENTIAL INCLUSIONS

Jean-Pierre Aubin

Introduction

Let $K \subset X$ be a closed subset of a Hilbert space X. We are looking for trajectories $x(\cdot)$ of dynamical systems satisfying the condition

(1) $$\forall t \in [0,T[\ , \quad x(t) \in K .$$

We shall call trajectories $x(\cdot)$ satisfying the latter property "vaible trajectories". The viability requirement is essential in many applications, where the state of a dynamical system has to obey several constraints when time evolves. This is obviously the case in economics (scarcity constraints), in mechanics (unilateral constraints) and in ecology (survival constraints).

Let us consider an ordinary differential equation $x' = f(x)$ where f is a continuous map from K to X. We introduce also the contingent cone $D_K(x)$ to K at x ; we recall that

(2) $$v \in D_K(x) \text{ if and only if } \liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0$$

(see Aubin [1]).

A necessary condition for this differential equation to have viable trajectories for every initial state $x_0 \in K$ is that

(3) $$\forall x \in K, \quad f(x) \in D_K(x) .$$

Indeed, $x'(0) = f(x_0)$ and, since $x(h) \in K$, $\frac{d_K(x_0+hx'(0))}{h} \leq \left\| x'(0) - \frac{x(h) - x_0}{h} \right\|$ converges to 0 with h. In 1942, Nagumo proved the converse statement:

Let $K \subset X$ be a closed locally compact subset and $f: K \to X$ be a continuous map satisfying the tangential condition (3). Then, for any $x_0 \in K$, there exists $T > 0$ such that the differential equation $x' = f(x)$ has a viable trajectory on $[0,T[$ starting at x_0. *Note that when $X = \mathbb{R}^n$, any closed subset is locally compact.*

This characterization of viability is actually quite natural, in the sense that it requires that for any boundary point x of K, the velocity $f(x)$, being "tangent" to K at x, forces the trajectory to bounce back in K.

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We also have to mention that when A is a maximal monotone set-valued map from $K \doteq D(A)$ to X, the single-valued map f defined by $f(x) \doteq -m(A(x))$ [where $m(L)$ is the element of L with minimal norm) satisfies condition (3). (See Brézis [1]).

This is no wonder for the differential inclusion

$$x' \in -A(x) \;;\quad x(0) = x_0$$

is equivalent to the differential equation

$$x' = -m(A(x)), \quad x(0) = x_0$$

which has viable trajectories $(x(t) \in D(A)$ for all $t \geq 0$) for all initial state $x_0 \in K$.

The first aim of this paper is to generalize the Nagumo theorem to the case of upper hemicontinuous maps from K to the closed convex subsets of X. This was done successfully by G. Haddad, who proved all the results that follow.

We mention for instance the basic statement: Let $K \subset X$ be a closed locally compact subset and F be a proper upper hemicontinuous map from K to the closed convex subsets of X. We posit the following tangential condition

$$(4) \qquad \forall x \in K, \quad F(x) \cap D_K(x) \neq \emptyset .$$

Then, for any $x_0 \in K$, there exists $T > 0$ such that the differential inclusion $x' \in F(x)$, $x(0) = x_0$ has a viable trajectory on $[0,T[$. If $F(K)$ is compact, we can take $T = \infty$.

Actually, the tangential condition is also necessary for the differential inclusion to have viable trajectories for every initial state.

When K is <u>convex and compact</u>, we can prove that the tangential condition (3) is not only a characterization of viability, but also a sufficient condition for the existence of <u>stationary points</u> (or <u>equilibria</u>) $x_* \in K$, i.e., solutions to the inclusion $0 \in F(x_*)$, or, states such that the constant function $t \mapsto x_* \in K$ is a trajectory. This is the Browder-Ky Fan theorem (see for instance Aubin [2], chapter 15).

Viability is a particular case of the problem of finding what we call monotone trajectories. We consider a set-valued map P from K to K satisfying the following properties

$$(5) \quad \begin{cases} \text{i)} & \forall x \in K, \quad x \in P(x) \quad \text{(reflexivity)} \\ \text{ii)} & \forall x \in K, \quad \forall y \in P(x), \text{ then } P(y) \subset P(x) \quad \text{(transitivity).} \end{cases}$$

In other words, the binary relation $\leq$ defined by

(6) $\qquad y \leq x$ if and only if $y \in P(x)$

is a preorder on K. A monotone trajectory is a function $x(\cdot)$ that satisfies

(7) $\qquad \forall s \geq t , \quad x(s) \in P(x(t)) \quad (\text{or} \quad x(s) \leq x(t))$.

This property is quite important in many problems. This is the case for instance in economics, where the preorder is the preorder of preference: $P(x)$ is the set of elements y preferred to x. When $P(x) = K$ for all $x \in K$, a monotone trajectory is nothing other than a viable trajectory. G. Haddad proved that if the set-valued map P is lower semicontinuous and has a closed graph, the tangential condition

(8) $\qquad \forall x \in K , \quad F(x) \cap D_{P(x)}(x) \neq \emptyset$

is a necessary and sufficient condition for the differential inclusion $x' \in F(x)$ to have monotone trajectories for all initial states $x_0 \in K$.

We treat in Aubin [1] the important case when the preorder is defined by a real-valued function

(9) $\qquad P(x) = \{y \in X \mid V(y) \leq V(x)\}$.

This particular case enjoys more properties, which are connected with the problem of stability of stationary points.

As usual, we shall deduce from the time-independent case the time dependent case.

Let $K: [0,\infty[\to X$ be a proper set-valued map with closed graph. A viable trajectory is defined in this case by

(10) $\qquad \forall t \in [0,T[, \quad x(t) \in K(t)$.

We introduce the contingent derivative $DK(t,x)(1)$ of $K(\cdot)$ at $(t,x) \in \text{graph}(K)$ in the direction 1: We recall that $v \in DK(t,x)(1)$ if and only if

(11) $$\liminf_{\substack{h \to 0+ \\ \tau \to 1}} d\left(v, \frac{K(t+h\tau)-x}{h}\right) = 0 .$$

So, we shall prove that if graph (K) is locally compact, if $F: \text{graph}(K) \to X$ is a proper upper hemicontinuous map with closed convex values that satisfy the condition

(12) $\qquad \forall t \geq 0 , \quad \forall x \in K(t), \quad F(t,x) \cap DK(t,x)(1) \neq \emptyset$,

then, for all $t_0 \geq 0$ and $x_0 \in K(t_0)$, there exists $T > 0$ such that the differential

inclusion $x' \in F(t,x)$, $x(0) = x_0$ has a viable trajectory on $[t_0, t_0+T]$.

The condition (12) is also necessary.

In many problems (as in optimization theory, game theory, mathematical economics), the subset K is defined by constraints. For instance, when $L \subset X$, $M \subset Y$ and $A \in \mathcal{L}(X,Y)$, we define K by

$$K \doteq \{x \in L \text{ such that } A(x) \in M\}.$$

We only know that in general,

$$D_K(x) \subset D_L(x) \cap A^{-1} D_M(Ax).$$

We would like to have sufficient conditions that involve only the subsets L, M and the operator A, as, for instance, $\forall x \in K$, $D_L(x) \cap A^{-1} D_M(Ax) \cap F(x) \neq \emptyset$. When $D_K(x) \neq D_L(x) \cap A^{-1} D_M(Ax)$, we prove that the condition

$$\forall x \in L, \ \exists v \in F(x) \text{ such that}$$

$$\liminf_{h \to 0+} \frac{1}{h} \max(d_L(x+hv), d_M(A(x+hv) - d_M(Ax)) = 0$$

implies the existence of trajectories satisfying

$$\forall t \in [0,T[, \ x(t) \in L \text{ and } A(x(t)) \in M.$$

Applications of the results we have mentioned are many. For instance, we can solve a class of implicit differential equations (or inclusions). Let $f: K \times X \to X$ be a continuous map that is affine with respect to the last argument and $F: K \to X$ be an upper hemicontinuous map with closed convex values. We shall prove the existence of viable trajectories of the implicit differential inclusion

$$(13) \qquad f(x,x') \in F(x).$$

Systems theory provides dynamical systems of this nature. If $\mathbb{R}^n$ is the vector space of the "states" of the system, $\mathbb{R}^m$ the vector space of "observations" of the system, we assume that observation of the states of the system is made via a continuously differentiable map B from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Dynamical systems whose velocity depends not only upon the state of the system, but also upon the variations of observations of the state of the system, are of the following form

$$(14) \qquad x' \in F(x) + C \frac{d}{dt} B(x)$$

where $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. So, we shall provide a viability criterion for these dynamical systems.

Another example of implicit differential equation is provided by the "Newton method", which yields trajectories whose cluster points are stationary points. Namely, let f be a continuously differentiable map from a neighborhood of a compact subset K of $\mathbb{R}^n$ to $\mathbb{R}^m$. We consider the implicit differential equation

$$(15) \qquad \nabla f(x)(x') = -f(x) \ .$$

The viability criterion

$$(16) \qquad \forall\, x \in K \ , \ \exists\, v \in D_K(x) \ \text{ such that } \ \nabla f(x)v = -f(x)$$

is exactly a sufficient condition implying the existence of stationary point $x_* \in K$ of f (See Aubin [1]). Since solutions of this equation satisfy

$$(17) \qquad f(x(t)) = e^{-t} f(x(0))$$

it is clear that cluster points of a trajectory $x(\cdot)$ are stationary points of f.

This result may be extended to proper set-valued maps F from K to X. The "Newton method" is written in this case in the form

$$(18) \qquad -e^{-t} y_0 \in DF(x(t), e^{-t} y_0)(x'(t))$$

where $DF(x,y)$ is the contingent derivative of F at $x \in K$ and $y \in F(x)$. In order to have a concrete algorithm, we assume that there exists a map $g(x,y)$ such that

$$\forall\, x \in K \ , \ \ \forall\, y \in F(x), \ \ -y \in DF(x,y)(g(x,y)).$$

Hence the Newton method can be written:

$$x'(t) = g(x(t), e^{-t} y_0), \ \ x(0) = x_0 \ \text{ where } \ y_0 \in F(x_0).$$

We can devise variants of this "method".

OUTLINE

1. Viable trajectories
2. Monotone trajectories
3. The time dependent case
4. Viable trajectories on sets defined by constraints
5. Application: Implicit differential inclusions
6. The Newton method

1. Viable trajectories

Let K be a closed subset of a Hilbert space X and $D_K(x)$ denote the *contingent* cone to K at x. We shall prove that under reasonable assumptions, a necessary and sufficient condition for the differential inclusion

(1) $$x' \in F(x) \;,\quad x(0) = x_0 \text{ where } x_0 \text{ is given in } K$$

to have trajectories that remain in K is that the following tangential condition

(2) $$\forall x \in K \;,\quad F(x) \cap D_K(x) \neq \emptyset$$

holds true.

Note that tangential condition (2) can be written in the form

(3) $$\forall x \in K \;,\quad \exists v \in F(x) \text{ such that } \liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0$$

which is closer to the statement of Nagumo's theorem. Recall also that the tangential condition (2) is equivalent to

(4) $$\begin{cases} \forall x \in K \;,\; \exists v \in F(x), \; \exists \text{ sequences of elements } h_n > 0 \text{ converging to } 0 \\ \text{and } v_n \in X \text{ converging to } v \text{ such that } x + h_n v_n \in K \;. \end{cases}$$

For simplicity, we shall say that trajectories $x(\cdot)$ such that

(5) $$\forall t \in [0,T[\;,\quad x(t) \in K$$

are "viable trajectories".

Let us begin by proving a necessary condition for a differential inclusion to have viable trajectories.

Proposition 1 (Haddad)

Let F be a proper upper semicontinuous map from a subset $K \subset \mathbb{R}^n$ with compact convex values.

If for all $x_0 \in K$, there exist $T > 0$ and a viable trajectory on $[0,T[$ of the differential inclusion $x' \in F(x)$ starting at x_0, then the tangential condition

(2) $$\forall x \in K \;,\quad F(x) \cap D_K(x) \neq \emptyset$$

holds. ∎

Proof. It is a corollary of Proposition 2.1 below. ∎

The tangential condition is also sufficient.

Theorem 1 (Haddad)

Let K be a closed subset of X, F be a proper upper hemicontinuous map from K to X with compact convex values. We posit the tangential condition:

(2) $$\forall\, x \in K\,,\quad F(x) \cap D_K(x) \neq \emptyset\ .$$

a) Assume that K is locally compact. Then for all $x_0 \in K$, there exists $T > 0$ such that the differential inclusion $x' \in F(x)$, $x(0) = x_0$ has a viable trajectory defined on $[0,T[$.

b) Assume that either K is compact or that X is finite-dimensional and $F(K)$ is bounded. Then for all $x_0 \in K$ there exists a viable trajectory of the differential inclusion defined on $[0,\infty[$. ■

Proof. It is a corollary of Theorem 2.1 below. ■

Remark. When $X = \mathbb{R}^n$, any closed subset K is locally compact.

Naturally, when F is a single-valued map, we obtain the Nagumo theorem as a consequence.

2. Monotone trajectories

The problem of finding viable trajectories of a differential inclusion is a particular case of the problem of looking for "monotone trajectories".

Let $\leq$ be a preorder defined on K, i.e., a binary relation $x \leq y$ which is

$$(1) \quad \begin{cases} \text{i) reflexive (i.e., } \forall x \in K, \ x \leq x) \\ \text{ii) transitive (i.e., if } y \leq x \text{ and } z \leq y, \text{ then } z \leq x. \end{cases}$$

We shall say that a trajectory $x(\cdot)$ defined on $[0,T[$ is "monotone" if and only if

$$(2) \qquad \forall\, t,s \in [0,T[, \quad s \geq t, \quad \text{then} \quad x(s) \leq x(t).$$

It is convenient to characterize a preorder $\leq$ by the set-valued map P defined by

$$(3) \qquad y \in P(x) \quad \text{if and only if} \quad y \leq x.$$

Conversely, let P be a set-valued map satisfying

$$(4) \quad \begin{cases} \text{i)} & \forall x \in K, \ x \in P(x) \quad \text{(reflexivity)} \\ \text{ii)} & \forall x \in K, \ \forall y \in P(x), \text{ we have } P(y) \subset P(x) \quad \text{(transitivity)}. \end{cases}$$

Then the relation (3) defines a preorder. So, a trajectory $x(\cdot)$ is monotone if and only if

$$(5) \qquad \forall\, t,s \in [0,T[, \quad s \geq t, \quad \text{then} \quad x(s) \in P(x,(t)).$$

The typical example of a preorder is the one defined by m real-valued functions $V_j : K \to \mathbb{R}$ $(j=1,\ldots,m)$:

$$\forall x \in K, \quad P(x) = \{y \in K \mid \forall j=1,\ldots,m, \ V_j(y) \leq V_j(x)\}.$$

For this preorder, a trajectory $x(\cdot)$ is monotone if and only if

$$\forall j=1,\ldots,m, \quad \forall\, s,t \in [0,T[, \quad s \geq t, \quad \text{then} \quad V_j(x(s)) \leq V_j(x(t)).$$

We study this specific example in Aubin [1].

We shall need the following property.

Lemma 1

If P is lower semicontinuous, then the function $(x,y) \to d_{P(x)}(y)$ is upper semicontinuous. ∎

Proof.

Since P is lower semicontinuous at x_0, we can associate to all $\varepsilon > 0$ an $\eta > 0$ such that $\|x-x_0\| \leq \eta$ implies that $P(x_0) \subset P(x) + \varepsilon/2\, B$. Then, $d(y_0, P(x) + \varepsilon/2\, B) \leq d(y_0,P(x_0))$. We also note that $d(y_0,P(x)) \leq d(y_0,P(x) + \varepsilon/2\, B) + \varepsilon/2$ and that

$d(y,P(x)) \leq d(y_0,P(x)) + \|y-y_0\|$. Hence, when $\|x-x_0\| \leq \eta$ and $\|y-y_0\| \leq \varepsilon/2$, we obtain:

$$d(y,P(x)) \leq d(y_0,P(x_0)) + \varepsilon .$$

Consequently $(x,y) \mapsto d_{P(x)}(y)$ is upper semicontinuous at (x_0,y_0) . ■

Example

Lemma.

If K *is convex and if the* m *functions* V_j *are* continuous and strictly quasi-convex, then the set-valued map P defined by (3) is lower semicontinuous. ■

Proof. It is left as an exercise. ■

We shall prove that a necessary and sufficient condition for a differential inclusion to have monotone trajectories is that the following tangential condition

(6) $$\forall x \in K , \quad F(x) \cap D_{P(x)}(x) \neq \emptyset$$

holds true. Note that when P is the constant map defined by $P(x) = K$, monotone trajectories are viable trajectories.

Let us begin by proving the necessity.

Proposition (Haddad)

Let F be a proper upper semicontinuous map from a subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^n$ with compact convex values. Let $P:K \to K$ be a proper set-valued map with closed images satisfying $x \in P(x)$ for all $x \in K$.

Let us assume that for all $x_0 \in K$, there exist $T > 0$ and a monotone trajectory on $[0,T[$ of the differential inclusion. Then the tangential condition

(6) $$\forall x \in K , \quad F(x) \cap D_{P(x)}(x) \neq \emptyset$$

is satisfied. ■

Proof.

Let $x(\cdot)$ be a solution to

(7) $$x' \in F(x) \; ; \quad x(0) = x_0$$

such that $\forall s \geq t$, $x(s) \in P(x(t))$. Since F is u.s.c., we can associate to any $\varepsilon > 0$ an $\eta > 0$ such that, for all $\tau \leq \eta$, $F(x(\tau)) \subset F(x_0) + \varepsilon B$.

Therefore, since $F(x_0) + \varepsilon B$ is convex and compact and since

$$\frac{x(h) - x_0}{h} = \frac{1}{h}\int_0^h x'(\tau)d\tau ,$$

the mean-value theorem implies that

$$\frac{x(h) - x_0}{h} \in F(x_0) + \varepsilon B .$$

Hence, there exists a subsequence h_n converging to 0 such that $v_n \doteq \frac{x(h_n) - x_0}{h_n}$ converges to some $v_0 \in F(x_0) + \varepsilon B$. By letting $\varepsilon \to 0$, we deduce that $v_0 \in F(x_0)$. Because $x(h_n) = x_0 + h_n v_n$ belongs to $P(x_0)$, we conclude that v_0 also belongs to $D_{P(x_0)}(x_0)$. ∎

The tangential condition is also sufficient.

<u>Theorem 1</u> (Haddad)

Let K be a closed subset of X, F be a proper upper hemicontinuous map from K to X with compact convex values and $P:K \to K$ be a lower semicontinuous map with closed graph satisfying

$$(4)\quad \begin{cases} \text{i)} & \forall x \in K ,\ x \in P(x) \ \text{(reflexivity)} \\ \text{ii)} & \forall x \in K ,\ \forall y \in P(x) ,\ \text{then } P(y) \subset P(x) \ \text{(transitivity)}. \end{cases}$$

We posit the following tangential condition.

$$(6)\qquad \forall x \in K ,\ F(x) \cap D_{P(x)}(x) \neq \emptyset .$$

a). Assume that K is locally compact. Then for all $x_0 \in K$, there exists $T > 0$ such that the differential inclusion

$$(7)\qquad x' \in F(x) ,\ x(0) = x_0$$

has a solution on $[0,T[$ that satisfies

$$(5)\qquad \forall t, \forall s \geq t ,\ x(s) \in P(x(t)).$$

b). Assume that either K is compact or that X is finite-dimensional and $F(K)$ is bounded. Then there exists a solution to the problem (7), (5) which is defined on $[0,\infty[$. ∎

Proof.

a). Let K and $x_0 \in K$ be given.

α). If K is relatively compact, there exists $r > 0$ such that $K_0 \doteq K \cap (x_0 + rB)$ is compact. We set $T = r/(\|F(K_0)\| + 1)$.

β). If K is compact, we set $K_0 \doteq K$ and $T = \infty$.

γ). If $F(K)$ is bounded and X is finite dimensional we take $T > 0$ arbitrary and we set $K_0 \doteq K \cap c\ell(x_0 + B + TF(K))$.

b). For all $y \in K$, we can find $h_y < 1/k$ and $v_y \in F(y)$ satisfying, by the tangential assumption (6),

$$d_{P(y)}(y + h_y v_y) < h_y/2k \,. \tag{8}$$

We introduce the subsets

$$N(y) = \{x \in X \mid d_{P(x)}(x + h_y v_y) < h_y/2k\} \,. \tag{9}$$

By Lemma 1, the function $x \mapsto d_{P(x)}(x + h_y v_y)$ is upper semicontinuous. Hence the subsets $N(y)$ are open. Since $y \in N(y)$, by (8), there exists a ball $B(y,\eta_y)$ of radius $\eta_y < 1/k$ contained in $N(y)$. Therefore, the compact subset K_0 can be covered by q such balls $B(y_j,\eta_{y_j})$. For simplicity, we set $\eta_j \doteq \eta_{y_j}$, $h_j \doteq h_{y_j}$, $v_j \doteq v_{y_j}$ $(j=1,\dots,q)$ and $h_0(k) = \min\limits_{j=1,\dots,q} h_j > 0$.

c). Now, let $x \in K_0$ be fixed. It belongs to $B(y_j,\eta_j) \subset N(y_j)$ for some index j. Consequently, we can find $x_j \in P(x)$ such that

$$\|v_j - (x_j - x)/h_j\| \le \frac{1}{h_j} d_{P(x)}(x + h_j v_j) + \frac{1}{2k} \le \frac{1}{k} \,.$$

We set $u_j = (x_j - x)/h_j$. So, cancelling the index j, we have proved that for all $x \in K_0$, there exist $h \in [h_0(k), \frac{1}{k}]$ and $u \in X$ such that

$$\left\{\begin{array}{lll} \text{i)} & & x + hu \in P(x) \\ \text{ii)} & & \exists\; y \in K \text{ and } v \in F(y) \text{ such that } \|x-y\| \le \frac{1}{k}, \ \|u-v\| \le \frac{1}{k} \,. \end{array}\right. \tag{10}$$

d). Let $x_0 \in K$ be fixed. Then we can find $h_0 \in [h_0(k), \frac{1}{k}]$ and $u_0 \in X$ such that $x_1 \doteq x_0 + h_0 u_0 \in P(x_0)$ and such that $(x_0,u_0) \in$ graph $F + \frac{1}{k}$. Furthermore, $x_1 - x_0 \in h_0(F(x_0) + \frac{1}{k}B) \subset h_0(F(K_0) + B)$. Hence, $\|x_1 - x_0\| \le h_0(\|F(K_0)\| + 1)$. Therefore, in the cases α) and γ), $\|x_1 - x_0\| \le r$ when $h_0 \le T$ and thus, $x_1 \in K_0$.

In case β), x_1 belongs also to K_0.

Hence, there exist $h_1 \in [h_0(k), \frac{1}{k}]$ and $u_1 \in X$ such that $x_2 \doteq x_1 + h_1 u_1 \in P(x_1)$ and such that $(x_1, u_1) \in \text{graph } F + \frac{1}{k}(B \times B)$. Furthermore, $x_2 - x_0 \in (h_0+h_1)(F(K_0) + \frac{1}{k} B)$, and thus, $\|x_2 - x_0\| \leq (h_0+h_1)(\|F(K_0)\| + 1)$. Therefore, in the cases α) and γ), $\|x_2 - x_0\| \leq r$ when $h_0+h_1 < T$ and thus, $x_2 \in K_0$. In the case β), x_2 belongs to K_0.

Since the h_j's belong to $[h_0(k), \frac{1}{k}]$, we are sure that there exists an integer m such that

(11) $$h_0 + h_1 + \ldots + h_m \leq T < h_1 + \ldots + h_m + h_{m+1} .$$

In summary, we have constructed sequences of $h_p \in [h_0(k), \frac{1}{k}]$, of x_p's $\in K_0$ and of u_p's $\in X$ such that

(12) $$\begin{cases} \text{i)} & x_{p+1} \doteq x_p + h_p u_p \in P(x_p) \\ \text{ii)} & (x_p, u_p) \in \text{graph}(F) + \frac{1}{k}(B \times B) . \end{cases}$$

This sequence is finite in the cases α) *and* γ) *and infinite in the case* β).

e). Let us set $\tau_k^q = h_0 + \ldots + h_q$.

We interpolate this sequence by the piecewise linear function $x_k(\cdot)$ defined on each interval $]\tau_k^{p-1}, \tau_k^p]$ by

$$x_k(t) \doteq x_{p-1} + (t - \tau_k^{p-1}) u_{p-1} .$$

Let t be fixed in $]\tau_k^{p-1}, \tau_k^p]$. We have $|t - \tau_k^p| \leq \frac{1}{k}$ and there exists $(y_p, v_p) \in \text{graph}(F)$ such that $\|x_k'(t) - v_p\| = \|u_p - v_p\| \leq \frac{1}{k}$ and $\|x_k(t) - y_p\| \leq \|x_k(t) - x_p\| + \|x_p - y_p\| \leq |t - \tau_k^p|(\|v_p - v_p\| + \|v_p\|) + \|x_p - y_p\| \leq \frac{1}{k}(\frac{1}{k} + \|F(K_0)\| + 1) \leq \frac{1}{k}(\|F(K_0)\| + 2)$. We set $\tilde{F}(t,x) \doteq F(x)$. We have proved that $\forall\, t \geq 0$,

(13) $$(t, x_k(t), x_k'(t)) \in \text{graph}(\tilde{F}) + \varepsilon(k)(B \times B \times B)$$

where $\varepsilon(k) \to 0$ when $k \to \infty$. We also know that

(14) $$\|x_k'(t)\| \leq \|F(K_0)\| + 1$$

and that

(15) $$x_k(t) \in \text{co}(K_0), \text{ which is compact.}$$

Hence the assumptions of the Convergence Theorem (see Aubin-Cellina [1]) are satisfied. A subsequence (again denoted) $x_k(\cdot)$ converges to a solution $x(\cdot)$ of the differential inclusion uniformly over compact intervals.

f). It remains to check that the trajectory is monotone.

Let us choose $s > t$. For k large enough, we can find $p \geq q$ such that $\tau_k^p \geq \tau_k^q$ converge to s and t respectively. Then, thanks to the transitivity of P, inclusions $x_k(\tau_k^j) \in P(x_k(\tau_k^{j-1}))$ imply that $x_k(\tau_k^p) \in P(x_k(\tau_k^q))$, i.e., that $(x_k(\tau_k^q), x_k(\tau_k^p)) \in \text{graph}(P)$. Since the latter is closed and since $x_k(\tau_k^p)$ and $x_k(\tau_k^q)$ converge to $x(s)$ and $x(t)$ respectively, we deduce that $(x(t),x(s)) \in \text{graph}(P)$, i.e., that $x(s) \in P(x(t))$. ∎

The solution $x(\cdot)$ belongs to $C(0,T;X)$ in the cases α) and γ) and to $C(0,\infty;X)$ in the case β). Since T is chosen independently of x_0 in the case γ), we can extend the monotone trajectory $x(\cdot)$ defined on $[0,T]$ to a trajectory defined on $[0,2T]$, $[0,3T]$, and so on. Hence, there exists a monotone trajectory $x(\cdot) \in C(0,\infty;X)$ in the case γ).

3. The time dependent case

We shall study now time dependent differential inclusions

(1) $$x' \in F(t,x) \ ; \ x(0) = x_0 \ .$$

We shall look for time-dependent viable trajectories, i.e., trajectories $x(\cdot)$ defined on $[0,T[$ and satisfying:

(2) $$\forall t \in [0,T[\ , \ x(t) \in K(t)$$

where $t \mapsto K(t)$ is a proper set-valued map from $[0,T[$ to X.

Let us consider the contingent derivative $DK(t,x)$ of the set-valued map K, in particular, we recall that

(3) $$v \in DK(t,x)(1) \text{ if and only if } (1,v) \in D_{\text{graph}(K)}(t,x)$$

i.e., if and only if

(4) $$\liminf_{\substack{h \to 0+ \\ \tau \to 1}} d(v, \frac{K(t+h\tau)-x}{h}) = 0$$

(see Aubin [1]).

If K is locally Lipschitzean, it can be written in the simpler form

(5) $$\liminf_{h \to 0+} d(v, \frac{K(t+h)-x}{h}) = 0 \ .$$

We shall prove that the differential inclusion (1) has viable trajectories starting at $x_0 \in K(t_0)$ at time $t_0 > 0$ if and only if the condition

(6) $$\forall t \geq 0 \ , \ \forall x \in K(t), \ F(t,x) \cap DK(t,x)(1) \neq \emptyset$$

holds true.

As usual, statements concerning the time dependent case follow from analogous statements on the time independent case by using the following transformation.

We consider the graph $\hat{K} \subset \mathbb{R} \times X$ of the set-valued map $K(\cdot)$ and the set-valued map $\hat{F}$ from $\hat{K}$ to $\mathbb{R} \times X$ defined by

(7) $$\hat{F}(t,x) \stackrel{\bullet}{=} (1,F(t,x)) \ .$$

So the viable trajectories $\tau \mapsto (t(\tau),x(\tau))$ of the time-independent differential inclusion $(t,x)' \in \hat{F}(t,x)$, i.e.,

(8) $$\begin{cases} \text{i)} & \dfrac{dt(\tau)}{d\tau} = 1 \\ \text{ii)} & \dfrac{dt(\tau)}{d\tau} \in F(t(\tau),x(\tau)) \text{ for almost all } \tau \in [0,T] \end{cases}$$

starting at $(t_0,x_0) \in \hat{K}$ yield the trajectories of the time dependent differential inclusion

(9) $$x'(t) \in F(t,x), \quad x(t_0) = x_0 \text{ given in } K(t_0)$$

satisfying

(10) $$\forall\, t \in [t_0,t_0+T[, \quad x(t) \in K(t).$$

Also, we note that the tangential condition for the differential inclusion $(t,x)' \in F(t,x)$ is

(11) $$\forall\, (t,x) \in \hat{K} = \text{graph}(K), \quad (1,F(t,x)) \in D_{\text{graph}(K)}(t,x) .$$

By the very definition of the contingent derivative of K, this is equivalent to the condition (6).

So Proposition 1.1 and Theorem 1.1 imply the following statements.

Proposition 1

Let K be a proper set-valued map from $[0,\infty[$ to $\mathbb{R}^n$ and F be a proper bounded upper semicontinuous map from graph(K) to $\mathbb{R}^n$ with compact convex values. If for all $t_0 \geq 0$ and $x_0 \in K(t_0)$, there exist $T > 0$ and a viable trajectory on $[t_0,t_0+T[$ of the differential inclusion

(1) $$x' \in F(t,x), \quad x(t_0) = x_0$$

then the condition

(6) $$\forall\, t > 0, \quad \forall\, x \in K(t), \quad F(t,x) \cap DK(t,x)(1) \neq \emptyset$$

holds true. ∎

Theorem 1

Let K be a proper set-valued map from $[0,\infty[$ to a Hilbert space X with closed graph and F be a proper upper hemicontinuous map from graph(K) to $\mathbb{R} \times X$ with compact convex values. We posit the condition:

(6) $$\forall\, t > 0, \quad \forall\, x \in K(t), \quad F(t,x) \cap DK(t,x)(1) \neq \emptyset .$$

a) Assume that graph(K) is locally compact. Then for all $t_0 > 0$, for all $x_0 \in K(t_0)$, there exists $T > 0$ such that the differential inclusion (1) has a viable trajectory on $[t_0,t_0+T[$.

b) Assume that X is finite dimensional and that $F(\mathrm{graph}(K))$ is bounded. Then for all $t_0 > 0$ and for all $x_0 \in K(t_0)$, there exists a viable trajectory on $[t_0,\infty[$ of the differential inclusion (1). ∎

We can also, by the same device, characterize the existence of time-dependent monotone trajectories. We define the time-dependent preorder by a set-valued map from graph(K) to graph(K) satisfying:

$$(12) \quad \begin{cases} \text{i)} & \forall\, t > 0\,,\ \forall\, x \in K(t),\ (t,x) \in P(t,x) \quad \text{(reflexivity)} \\ \text{ii)} & \forall (s,y) \in P(t,x),\ \text{we have}\ P(s,y) \subset P(t,x) \quad \text{(transitivity)}. \end{cases}$$

So, for each $(t,x) \in \mathrm{graph}(K)$, $P(t,x)$ can be regarded as the graph of a set-valued map $Q_{t,x}(\cdot)$ from $[t,\infty[$ to X defined by

$$(13) \qquad \forall\, s \geq t,\ \ y \in Q_{t,x}(s) \ \text{ if } \ (s,y) \in P(t,x).$$

In other words if $P(t,x)$ is regarded as defining a "preference preorder", then $Q_{t,x}(s)$ is the subset of elements y which are "preferred at time s to an element x given at time t". Therefore, if F is a set-valued map defined on graph(K), to say that

$$(14) \qquad \forall\, t > 0\,,\ \forall\, x \in K(t),\ (1,F(t,x)) \cap D_{P(t,x)}(t,x) \neq \emptyset$$

amounts to saying that

$$(15) \qquad \forall\, t > 0\,,\ \forall\, x \in K(t),\ F(t,x) \cap DQ_{t,x}(t,x)(1) \neq \emptyset\,.$$

This condition is necessary and sufficient for the existence of time dependent monotone trajectories. We only state the following consequence of Theorem 1.1.

Theorem 2

Let K be a proper set-valued map from $[0,\infty[$ to a Hilbert space X with closed graph, $P:\mathrm{graph}(K) \to \mathrm{graph}(K)$ be a proper lower semicontinuous map with closed graph satisfying reflexivity and transitivity conditions (12) and $F:\mathrm{graph}(K) \to \mathbb{R} \times X$ be a proper upper hemicontinuous map with compact convex values. We posit the following condition:

$$(15) \qquad \forall\, t > 0\,,\ \forall\, x \in K(t),\ F(t,x) \cap DQ_{t,x}(t,x)(1) \neq \emptyset\,.$$

a) Assume that graph(K) is locally compact. Then, for all $t_0 > 0$ and $x_0 \in K(t_0)$, there exists $T > 0$ such that the differential inclusion

(1) $$x' \in F(t,x), \quad x(0) = x_0$$

has a solution on $[t_0,t_0+T[$ that is *monotone* in the sense that

(16) $$\forall\, s,t \in [t_0,t_0+T[, \quad s \le t, \quad x(s) \in Q_{t,x(t)}(s) .$$

b) Assume that X is finite dimensional and that F is bounded. We can take $T = \infty$. ■

4. Viable trajectories on sets defined by constraints.

We consider the case when the subset K is defined by constraints. For instance, if $A \in \mathcal{L}(X,Y)$ is a continuous linear operator, if $L \subset X$ and $M \subset Y$ are nonempty subsets, we define K by

(1) $$K \doteq \{x \in L \mid Ax \in M\} = L \cap A^{-1}(M).$$

We would like to express the tangential condition

(2) $$\forall x \in K, \quad F(x) \cap D_K(x) \neq \emptyset$$

in terms of L,M and A.

If we were in the favorable situation when

(3) $$\forall x \in K, \quad D_K(x) = D_L(x) \cap A^{-1} D_M(Ax),$$

the tangential condition is indeed couched in terms involving L,M and A:

(4) $$\forall x \in K, \quad F(x) \cap D_L(x) \cap A^{-1} D_M(Ax) \neq \emptyset .$$

We recall that property (3) holds when, for instance, L and M are closed convex subsets satisfying $0 \in \mathrm{Int}(A(L) - M)$. (see Aubin [3]).

In general, only the inclusion $D_K(x) \subset D_L(x) \cap A^{-1} D_M(Ax)$ holds, so that condition (4) does not imply necessarily the tangential condition (3).

Hence we devise other sufficient conditions for the existence of viable trajectories.

For instance, in the case when $K \doteq L \cap A^{-1}(M)$, we may assume that

(5) $$\begin{cases} \forall x \in L, \ \exists v \in F(x) \text{ such that} \\ \liminf_{h \to 0+} \frac{1}{h} \max\{d_L(x+hv), d_M(Ax+h\,Av) - d_M(Ax)\} = 0 . \end{cases}$$

Theorem 1.

Let $L \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ be two closed subsets and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Let F be a proper upper hemicontinuous map from L to the compact convex subsets of $\mathbb{R}^n$. We posit the above assumption (5).

Then for all $x_0 \in L$ satisfying $A(x_0) \in M$, there exists $T > 0$ such that the differential inclusion $x' \in F(x)$, $x(0) = x_0$ has a solution on $[0,T[$ satisfying

(5) $$\forall t \in [0,T[, \quad x(t) \in L \text{ and } Ax(t) \in M .$$

If $F(L)$ is bounded, we can take $T = +\infty$. ∎

Proof.

It is largely similar to the proof of Theorem 2.1.

a) We set $L_0 \doteq L \cap (x_0 + rB)$ and $T \doteq r/(\|F(L_0)\| + 1)$ or T arbitrary and $L_0 = L \cap (c\ell(x_0 + T\,F(L) + B))$ when $F(L)$ is bounded.

b) We associate with any $y \in L$ an element $v_y \in X$ and $h_y < \frac{1}{k}$ satisfying

(5) $$\max\{d_L(y+h_y v_y),\ d_M(Ay+h_y\,Av_y) - d_M(Ay)\} < h_y/2k$$

and we introduce the open subset

(6) $$N(y) \doteq \{x \in X \mid \max\{d_L(x+h_y v_y),\ d_M(Ax+h_y v_y) - d_M(Ax)\} < h_y/2k\} .$$

Since $y \in N(y)$, we can find a ball $B(y,\eta_y)$ with radius $\eta_y < 1/k$ contained in $N(y)$. Therefore, the compact subset L_0 can be covered by q such balls $B(y_j,\eta_{y_j})$. We set $\eta_j \doteq \eta_{y_j}$, $h_j \doteq h_{y_j}$, $v_j \doteq v_{y_j}$ and $h_0(k) = \min_{j=1,\dots,q} h_j > 0$.

c) Now, let $x \in L_0$ be chosen. It belongs to some $B(y_j,\eta_j) \subset N(y_j)$. Consequently, we can find $x_j \in L_0$ such that

$$\left\| v_j - \frac{x_j - x}{h_j} \right\| \le \frac{1}{h_j}\ d_L(x + h_j\ v_j) + \frac{1}{2k} \le \frac{1}{k} .$$

Hence, if we set $u_j \doteq \frac{x_j - x}{h_j}$, we obtain

$$d_M(Ax + h_j\ Au_j) \le d_M(Ax\ + h_j\ A\ v_j)\ + h_j\ \|A\|\|u_j - v_j\|$$

and since $x \in N(y_j)$,

$$d_M(Ax + h_j\ Au_j) \le d_M(Ax) + h_j/2k + h_j\|A\|\|u_j - v_j\| \le d_M(Ax) + (\|A\| + \tfrac{1}{2})\ h_j/k$$

So, cancelling the index j, we have proved that for all $x \in L$, $\exists\, h \in [h_0(k), \frac{1}{k}]$ and $u \in X$ such that

(7) $$\begin{cases} \text{i)} & x + hu \in L \\ \text{ii)} & d_M(Ax + h\ Au) - d_M(Ax) \le (\|A\| + \frac{1}{2})\ h/k \\ \text{iii)} & (x,u) \in \mathrm{graph}(F) + \frac{1}{k}\,(B\times B). \end{cases}$$

d) Therefore, we can construct sequences of scalars $h_p \in [h_0(k), \frac{1}{k}]$, of elements $x_p \in L_0$ and of elements $u_p \in X$ such that

$$
(8) \quad \begin{cases} \text{i)} & x_{p+1} \doteq x_p + h_p u_p \in L \text{ for all } p = 0,1,\ldots \\ \text{ii)} & d_M(Ax_{p+1}) - d_M(Ax_p) \leq (\|A\| + \frac{1}{2})h_p/k \\ \text{iii)} & (x_p, u_p) \in \text{graph}(F) + \frac{1}{k}(B\times B) \end{cases}
$$

e) We set $\tau_k^q = h_0 + \ldots + h_q$. We interpolate this sequence by the piecewise linear functions $x_k(\cdot)$ defined on each interval $]\tau_k^{p-1}, \tau_k^p]$ by $x_k(t) \doteq x_{p-1} + (t - \tau_k^{p-1})u_{p-1}$. As in the proof of Theorem 2.1, we deduce from the Convergence Theorem (see Aubin-Cellina [1]) that a subsequence of $x_k(\cdot)$ converges to a solution $x(\cdot)$ of the differential inclusion satisfying

$$(9) \qquad \forall\, t \in [0,T[\,,\quad x(t) \in L\,.$$

Since the approximate solutions $x_k(\cdot)$ satisfy, thanks to (8) ii),

$$(10) \qquad \forall\, p = 0,\ldots,\quad d_M(A\, x_k(\tau_k^{p+1})) - d_M(A\, x_k(\tau_k^p)) \leq (\|A\| + \frac{1}{2})h_p/k\,,$$

we deduce, by summing up these inequalities from $p = 0$ to $p = q-1$ and using the fact that $d_M(A\, x(0)) = d_M(A\, x_0) = 0$ for $x_0 \in K \doteq L \cap A^{-1}(M)$, that

$$(11) \qquad d_M(A\, x_k(\tau_k^q)) \leq (\|A\| + \frac{1}{2})\tau_k^q/k\,.$$

Since any $t \in [0,T[$ can be approximated by suitable τ_k^q and since $x_k(\tau_k^q)$ converges to $x(t)$, it follows that $d_M(Ax(t)) = 0$, i.e., that $A\,x(t) \in M$. Hence, $\forall\, t \in [0,T[$, $Ax(t) \in M$. ∎

5. Application: Implicit differential inclusions.

Let f be a single-valued map from $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$. We shall solve implicit differential inclusions.

(1) $\quad f(t,x(t), x'(t)) \in F(t,x(t));\; x(0) = x_0$ is given in K.

Theorem 1

We assume that the graph K of the set-valued map $t \in \mathbb{R}_+ \to K(t) \subset \mathbb{R}^n$ is closed, that the map $f:(t,x,v) \in K \times \mathbb{R}^n \to f(t,x,v) \in \mathbb{R}^m$ is continuous and affine with respect to v and that the proper set-valued map $F:K \to \mathbb{R}^m$ is upper hemicontinuous with closed convex images. We posit the following tangential condition: There exists a constant $c > 0$ such that

$$(2) \quad \begin{cases} \forall\, t \geq 0\,,\ \forall\, x \in K(t),\ \exists\, v \in DK(t,x)(1) \cap cB \text{ such that} \\ f(t,x,v) \in F(t,x). \end{cases}$$

Then, for all $x_0 \in K(0)$, there exist a viable solution $x(\cdot)$ to the differential inclusion (1). ∎

Proof.

We set $G(t,x) = \{v \in \mathbb{R}^n \text{ such that } f(t,x,v) \in F(x)\}$. Since f is continuous and since the graph of F is closed, then obviously, the graph of G is also closed. Therefore, the set-valued map $H:(t,x) \mapsto G(t,x) \cap cB$ has nonempty values (by assumption (2)), is upper semicontinuous (having a closed graph and taking its values in the compact set cB, is bounded and has compact convex values. It also satisfies the condition:

$$\forall\,(t,x) \in K\,,\ \exists\, v \in H(t,x) \cap DK(t,x)(1).$$

Hence Theorem 3.1 implies the existence of viable solutions to the differential inclusion $x'(t) \in H(t,x(t)),\ x(0) = x_0$, which are obviously viable solutions to the implicit differential inclusion (1). ∎

Example.

Let $X = \mathbb{R}^n$ be the space of states of the system we wish to describe and $Y = \mathbb{R}^m$ be the space of "observations". We denote by $B : X \to Y$ the "observation map" of the system and by $C:Y \to X$ the "feedback map".

In this model, we assume that the evolution law is

(3) $$x'(t) \in F(x(t)) + C \frac{d}{dt} B(x(t)); \quad x(0) = x_0 .$$

In other words, we assume that the velocity depends not only upon the state of the system but also upon the variations of observations of the state.

We assume that

(4) $$\begin{cases} \text{i)} & C \in \mathcal{L}(Y,X) \text{ is continuous and linear} \\ \text{ii)} & B : X \to Y \text{ is continuously differentiable on an open subset} \\ & \Omega \text{ containing } K . \end{cases}$$

We set

(5) $$A(x) = 1 - C \nabla B(x) .$$

So, system (3) can be written

(6) $$A(x(t))\, x'(t) \in F(x(t)) .$$

<u>Proposition 1</u>

Assume that $K \subset X = \mathbb{R}^n$ is a closed subset, F is an upper hemicontinuous set-valued map from K into X with nonempty closed convex values. Let $C \in \mathcal{L}(Y,X)$ and $B \in C^1(\Omega,Y)$. We suppose that there exists $c > 0$ such that,

(7) $$\forall x \in K , \ \exists v \in F(x) + C \nabla B(x)v \ \text{ such that } \ v \in D_K(x) \cap cB .$$

Then, for any initial state $x_0 \in K$, there exists a viable solution to the inclusion (3).

■

6. The Newton method

We proved in Corollary 9.1 of Aubin [1] that when F is a continuously differentiable map from a neighborhood of a compact subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies the condition

(1) $$\forall x \in K, \quad \exists v \in D_K(x) \text{ such that } \nabla F(x)v = -F(x),$$

then there exists a stationary point $x_* \in K$ of F.

These assumptions imply also that there exist trajectories $x(\cdot)$ of the differential inclusion

(2) $$\nabla F(x)x' = -F(x), \quad x(0) = x_0$$

that remain in K and that converge to a stationary point of F then $t \to \infty$. (see Haddad [1]).

We can consider such trajectories as the continuous analogs of the classical <u>Newton method</u>, which yields the discrete trajectory defined recursively by

(3) $$\nabla F(x_n)(x_{n+1} - x_n) = -F(x_n) \; ; \; x_0 \text{ is given.}$$

<u>Theorem</u> 1

Let F be a continuously differentiable map from a neighborhood of closed subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^m$ satisfying

(4) $$\begin{cases} \exists c > 0 \text{ such that, } \forall x \in K, \quad \exists v \in D_K(x) \cap cB \text{ satisfying} \\ \nabla F(x)v = -F(x). \end{cases}$$

Then there exists a viable trajectory of the implicit differential equation (2) that satisfies

(5) $$F(x(t)) = e^{-t}F(x(0)).$$

Thus the cluster points of $x(t)$ (if any) are stationary points of F. ∎

<u>Proof</u>.

We set $G(x) \doteq -\nabla F(x)^{-1}F(x)$. Trajectories of the differential inclusion $x' \in G(x)$ are the trajectories of the implicit differential equation (2).

Assumptions of <u>Theorem</u> 5.1 on implicit differential inclusions are satisfied. Hence there exist viable trajectories of the implicit differential equation (2). Consider any such trajectory. Then, since F is continuously differentiable, we have

(6) $$\frac{d}{dt} F(x(t)) = \nabla F(x(t)x'(t) = -F(x(t)).$$

So $y(t) \doteq F(x(t))$ is equal to $e^{-t}y(0)$. Therefore, $F(x(t))$ converges to 0 when $t \to \infty$. Any cluster point $x_* \in K$, limit of a subsequence $x(t_n)$ when $t_n \to \infty$, satisfies $F(x_*) = \lim_{t_h \to \infty} F(x(t_n)) = 0$, and thus, is a stationary point of F. ∎

Recall that the above sufficient condition for existence of a stationary point of F can be extended to set-valued maps (see Theorem 9.1 of Aubin [1]): We replace the tangential condition (4) by

(7) $$\forall x \in K\ ,\ \exists\ y \in F(x),\ \exists\ u \in X \text{ such that } -y \in DF(x,y)(u)\ .$$

where $DF(x,y)$ is the contingent derivative of the set-valued map F.

We can generalize the Newton method if we assume, for instance, that

(8) $$\begin{cases} \text{there exists a bounded upper hemicontinuous proper set-valued map } G \\ \text{from graph}(F) \text{ to the closed convex subsets of } \mathbb{R}^n \text{ such that} \\ \forall x \in K\ ,\ \forall y \in F(x),\ \exists\ u \in G(x,y) \text{ such that } -y \in DF(x,y)(u)\ . \end{cases}$$

<u>Theorem</u> 2 (Saint-Pierre)

Let F be a proper map from $K \subset \mathbb{R}^n$ to $\mathbb{R}^m$ with closed graph. We posit Assumption (6). Then, for any $x_0 \in K$ and $y_0 \in F(x_0)$, there exists a solution to the differential inclusion

(9) $$x'(t) \in G(x(t), e^{-t}y_0)$$

that satisfies

(10) $$\forall t \geq 0\ ,\ x(t) \in K \text{ and } e^{-t}y_0 \in F(x(t)).$$

Thus the cluster points of $x(t)$ (if any) are stationary points of F. ∎

<u>Proof</u>

We consider the differential inclusion

(11) $$x' \in G(x,y),\quad y' = -y$$

with the initial condition $x(0) = x_0$, $y(0) = y_0$.

Condition (6) implies that

(12) $$\forall\ (x,y) \in \text{graph}(F),\quad (G(x,y),-y) \cap D_{\text{graph}(F)}(x,y) \neq \emptyset\ .$$

Then Haddad's theorem implies that there exists a trajectory $(x(t),y(t))$ of this differential inclusion which remains in graph(F). Furthermore, $y(t) = e^{-t}y_0$. Hence $e^{-t}y_0 \in F(x(t))$. The rest of the theorem ensues. ∎

Remark

We note that we can devise a whole family of algorithms that converge to a stationary point of F. Let H be any map from $\mathbb{R}^n$ to itself such that

$$(13)\qquad \begin{cases} \text{the solution of } y' = H(y),\ y(0) = y_0 \text{ is unique} \\ \text{and converges to } 0 \text{ when } t \to \infty . \end{cases}$$

We associate with such a map H a bounded continuous map G (single-valued for the sake of simplicity) such that

$$(14)\qquad \forall\ (x,y) \in \operatorname{graph}(F),\quad H(y) \in DF(x,y)(G(x,y)).$$

Then there exist solutions to the differential equation

$$(15)\qquad x'(t) \in G(x(t),y(t)),\quad y'(t) = H(y(t));\quad x(0) = x_0,\ y(0) = y_0$$

such that

$$(16)\qquad \forall\ t \geq 0\ ,\quad y(t) \in F(x(t))\ .$$

Since $\lim_{t\to\infty} y(t) = 0$ by assumption, the cluster points of $x(t)$ (if any) are stationary points of F. ∎

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This is an exposition of Haddad's theorems on the existence of solutions to differential inclusions $x' \in F(t,x)$, $x(t_0) = x_0$ that satisfy the "viability condition"

$$\forall\, t \in [0,T],\ x(t) \in K(t)$$

where the subsets $K(t)$ are given. The existence theorem generalizes the Nagumo theorem when F is single-valued and when both F and K do not depend on t. Haddad's theorems provide necessary and sufficient conditions for a differential inclusion to have viable trajectories. They involve the concepts of Bouligand

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20. Abstract (continued)

contingent cone to K when $K(t) = K$ does not depend on K and of contingent derivatives of the set-valued map $t \in K(t)$. We adapt these results to characterize the existence of monotone trajectories with respect to a preorder $\geq$:

$$\forall\, s > t\ ,\quad x(s) \geq x(t).$$

We conclude by presenting Saint-Pierre's theorem on the convergence of a generalized Newton method converging to a solution $u_* \in K$ of the inclusion $0 \in F(x_*)$ in a nondeterministic way. This is of upmost importance in economics where uncertainty plays a crucial role. It is also important in control theory (since the differential inclusion (5) can be regarded as involving implicitly controls) as well as for solving implicit differential equations $f(x,x') = 0$.